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Ergoregions in Magnetised Black Hole Spacetimes

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ABSTRACT

The spacetimes obtained by Ernst's procedure for appending an external magnetic field B to a seed Kerr-Newman black hole are commonly believed to be asymptotic to the static Melvin solution. We show that this is not in general true. Unless the electric charge of the black hole satisfies $Q = jB(1+\frac{1}{4}j^2B^4)$, where j is the angular momentum of the original seed solution, an ergoregion extends all the way from the black hole horizon to infinity. We give a self-contained account of the solution-generating procedure, including including explicit formulae for the metric and the vector potential. In the case when $Q = jB(1 + \frac{1}{4}j^2B^4)$, we show that there is an arbitrariness in the choice of asymptotically timelike Killing field $K_\Omega = \partial/\partial t + \Omega\, \partial/\partial\phi$, because there is no canonical choice of Ω. For one choice, $\Omega = \Omega_s$, the metric is asymptotically static, and there is an ergoregion confined to the neighbourhood of the horizon. On the other hand, by choosing $\Omega = \Omega_H$, so that K_{Ω_H} is co-rotating with the horizon, then for sufficiently large B numerical studies indicate there is no ergoregion at all. For smaller values, in a range $B_- < B < B_+$, there is a toroidal ergoregion outside and disjoint from the horizon. If $B \le B_-$ this ergoregion expands all the way to infinity in a cylindrical region near to the rotation axis. For black holes whose size is small compared to the Melvin radius $2/B$, we recover Wald's result that it is energetically favourable for the hole to acquire a charge $2jB$.

Contents

1 Introduction

Understanding the energetics of astrophysical black holes involves understanding their interactions with charged particles and with external magnetic fields. This has been the subject of many studies, going back to the early work of Wald [1], King, Kundt and Lasota [3], Blandford and Znajek [5], etc. (see [6, 7, 8] for recent reviews). In particular, when the black hole is rotating, the resulting "dragging"of the magnetic field induces electric fields which may have dramatic effects on charged particles, to the extent that it becomes energetically favourable for an initially neutral black hole of mass m and angular momentum j to to acquire a charge [1], and for currents to flow [5].

For all practical astrophysical purposes, the gravitational back-reaction of the magnetic field may be neglected, and the electromagnetic field may be treated as a "test"field on the unperturbed, asymptotically flat, electrically neutral Kerr solution, with mass parameter

m and angular momentum factor $j = am$. If the electromagnetic field is assumed to be stationary, one may then use an old result of Papapetrou [2] to show that the vector potential takes the form [1]

$$A = \Big(\frac{jB}{m} - \frac{Q}{2m}\Big)K_\flat + \tfrac{1}{2}B\mathfrak{m}_\flat\,, \tag{1.1}$$

where $K_\flat = K^\mu g_{\mu\nu}dx^\nu$ and $\mathfrak{m}_\flat = \mathfrak{m}^\mu g_{\mu\nu}dx^\nu$. Here, $K^\mu\partial_\mu = \frac{\partial}{\partial t}$ and $\mathfrak{m}^\mu\partial_\mu = \frac{\partial}{\partial\phi}$ are the time-translation and rotational Killing fields respectively, the constant B is the strength of the asymptotic magnetic field, and the constant Q is the charge inside the horizon. The electrostatic potential difference Φ_H, or "injection energy" between the black hole horizon and infinity is given by

$$\Phi_H = \frac{Q - 2jB}{2m} \tag{1.2}$$

and will vanish if the hole acquires the *Wald charge*

$$Q = 2jB\,. \tag{1.3}$$

The mechanism for the current flow required to lower the energy might be conduction through the ambient plasma, or a breakdown of the vacuum through pair production. A discussion of pair production from the point of view of black hole thermodynamics and quantum field theory was given long ago [4]; however, back-reaction was not then taken into account. More recently, Wald's original argument has been criticised by Li [9], who proposed a different value for the charge which minimizes the electromagnetic energy.

Since those early investigations, despite the availability of apparently appropriate exact solutions of the Einstein-Maxwell equations taking back reaction into account [10, 11], and an analysis of their properties [12, 13, 14, 15, 16, 17], no full treatment of black hole thermodynamics in the rotating case has yet been given. An appealing analogy to a Faraday disc, adopted by Blandford and Znajek, based on ideas of Damour [18], views the horizon as an electrical conductor with surface conductivity of 4π Ohms. This was elaborated upon by Thorne et al., under the rubric of the "Membrane Paradigm"[19]. It suggests that a full treatment, taking into account back-reaction and in particular the torque exerted by the rotating black hole on the source of the magnetic field, might be extremely rich. This expectation gains support from the striking fact that in the only case that has to date been studied exactly, namely that of a Schwarzschild black hole immersed in a background Melvin solution [20] (possibly with a dilaton in addition), it was found that the black hole thermodynamics was unaffected by the presence of the magnetic field, with both the entropy and temperature being unchanged [21]. There is a clear implication of this result that the microscopic degrees of freedom of the hole that are responsible for the entropy are unaffected

by the external magnetic field. The obvious question arises as to whether this remains true in the rotating case.

The reason for the absence of a full treatment is the complexity of the exact solutions that appear to be appropriate in this case, all of which have been obtained by means of Harrison type solution-generating techniques [22] starting from an initial Kerr-Newman metric. The default assumption in the literature has been that this will produce a background at infinity that is "asymptotically Melvin." If this were the case, it should then be a straightforward task to read off the total mass and angular momentum, calculate the electrostatic potentials, and hence get a handle on the generalised first law, possibly using Komar identities. So far, the complexity of the solutions, even at infinity, and possibly the absence of a direct astrophysical motivation, has prevented this programme being carried out. In this paper we shall show that there is a more serious obstruction: the relevant solutions turn out in general not to be asymptotic to the static Melvin solution. In fact, unless the charge parameter q of the magnetised Kerr-Newman solution is chosen to be $q = -amB$, where a is the rotation parameter, m the mass parameter and B the external magnetic field, they contain an ergoregion that extends out to infinity, close to, but not containing, the rotation axis, with timelike boundary. In other words, unless $q = -amB$, *the dragging of inertial frames is so strong that even at infinity there is no Killing vector field which is everywhere timelike outside a compact set containing the black hole Killing horizon.*

The criterion $q = -amB$ that the ergoregion does not extend to infinity may be re-expressed in terms of the total electric charge Q and the angular momentum $j = am$ of the original seed solution:

$$Q = jB(1 + \tfrac{1}{4}j^2B^4)\,. \tag{1.4}$$

Note that the quantity j should be distinguished from any measure of the total angular momentum of the magnetised spacetime. For asymptotically flat spacetimes, the total angular momentum J may be expressed as a Komar integral

$$J = \frac{1}{16\pi}\int *d\mathfrak{m}_\flat\,, \tag{1.5}$$

taken over a large sphere at spatial infinity. For a vacuum spacetime, such integrals do not depend on the choice of surface on which they are evaluated, provided the surface is homologous to the sphere at infinity. In the presence of an electromagnetic field, the Komar integral may be surface dependent. For example, in the case of a Kerr-Newman black hole, the total angular momentum j is given by a Komar integral over a surface at infinity, and this differs from the integral over the horizon because of the angular momentum carried

by the electromagnetic field outside the horizon. In the case of the magnetised spacetimes, difficulties arise in trying to evaluate (1.5) because of the asymptotic structure.

One may check that the existence of an ergoregion that extends to infinity, in the case that (1.4) is not satisfied, is independent of the choice of timelike Killing vector. More specifically, if one replaces $K = \partial/\partial t$ by $K_\Omega = \partial/\partial t + \Omega\partial/\partial\phi$, where Ω is a constant, then K_Ω will still be spacelike at infinity, close to the rotation axis.

If the charge does take the special value given by $q = -amB$ (or, equivalently, (1.4)), we find that there exists a range of choices for Ω, of the form $\Omega_- < \Omega < \Omega_+$, for which the Killing vector K_Ω is timelike everywhere at large distances. Within this range lies an angular velocity Ω_s for which the magnetised black hole metric is asymptotically static. In this frame, K_Ω becomes spacelike in a compact neighbourhood of the horizon, signaling the occurrence of an ergoregion that is similar to the one outside a standard Kerr or Kerr-Newman black hole. For another choice of Ω within the range, namely $\Omega = \Omega_H$, the angular velocity of the horizon, the Killing vector K_Ω is null on the horizon and, for sufficiently large B (greater than a certain critical value B_+), numerical studies indicate it is timelike everywhere outside the horizon. Thus in this frame, there is no ergoregion at all when $B > B_+$. If B lies in the range $B_- < B < B_+$, where B_- is another computable value of the magnetic field, there is an ergoregion of toroidal topology, outside the horizon and disjoint from it, lying in the equatorial plane. As B approaches B_- from above, the toroidal ergoregion extends upwards and downwards further and further, eventually reaching infinity if $B \le B_-$.

The possibility of making different choices for the asymptotically timelike Killing vector that generates "time translations" is something that does not arise in asymptotically flat stationary spacetimes, where there is a unique asymptotically timelike Killing vector. We include in this paper a detailed discussion of this phenomenon in asymptotically Melvin spacetimes, and make a comparison with the somewhat analogous situation that arises for stationary black holes in asymptotically AdS spacetimes.

The organisation of this paper is as follows. As a preliminary step, before considering the full magnetised Kerr-Newman solution, in section 2 we examine the much simpler case of the magnetised Reissner-Nordström black hole. This example is useful because it illustrates, in a simpler setting, the same problem that arises for a general magnetised Kerr-Newman black hole, namely, that there is no choice of Killing vector that is asymptotically timelike in all directions at infinity. Specifically, we find that near to the z axis any Killing vector of the form $K = \partial/\partial t + \Omega\,\partial/\partial\phi$ becomes spacelike, thus indicating the existence of an er-

goregion that extends to infinity. This also means that the magnetised Reissner-Nordström solution is not asymptotic to the Melvin solution. Only by setting the charge parameter to zero, so that the solution reduces to the Schwarzschild-Melvin metric, are these problems avoided. Section 2 also contains a brief discussion of the magnetisation of a *magnetically* charged Reissner-Nordström black hole. In section 3, we turn to the analysis of the magnetised Kerr-Newman solution. We show that for generic values of the mass, charge and rotation parameters m, q and a of the original seed Kerr-Newman solution, the metric again necessarily has an ergoregion that extends to infinity, and it is not asymptotic to the Melvin solution. We then show that this problem is avoided if the parameters obey the relation $q = -amB$. As we discuss in detail, the metric *is* now asymptotic to the Melvin metric, and we show how, depending on the choice of time-translation Killing vector, and the parameters of the solution, there can be either a compact ergoregion in the neighbourhood of the horizon, or a toroidal ergoregion outside the black hole, or else no ergoregion at all. In section 4 we make a comparison, highlighting the similarities and the differences, between the asymptotically Melvin black holes of this paper and the somewhat analogous case of black holes in an asymptotically AdS background. In section 5 we discuss the relation between our results for the exact magnetised black hole solutions and the earlier results of Wald, where the back-reaction of the external magnetic field on the geometry is neglected. Our conclusions, and a discussion of open problems, are in section 6. In appendix A we give an explicit dimensional reduction of four-dimensional Einstein-Maxwell theory to three dimensions, showing how it gives rise to an $SU(2,1)/U(2)$ sigma model coupled to gravity. We use the $SU(2,1)$ global symmetry in appendix B to obtain the complete expressions for the magnetised Kerr-Newman black hole solution, including both the metric and the vector potential. Finally, in appendix C, we show how another $SU(2,1)$ transformation, applied to a flat space "seed solution," gives rise to a simple cosmological metric first obtained by Taub, which exhibits some features that are rather similar to those we encountered for the magnetised Kerr-Newman metrics.

2 Magnetised Reissner-Nordström Black Hole

The general results for the magnetisation of the Kerr-Newman solution are obtained in appendices A and B. The expressions are quite complicated, and so before examining the global properties of the magnetised Kerr-Newman metrics in section 3, we first specialise to the simpler case where the rotation parameter a is set to zero.

2.1 Magnetised electric Reissner-Nordström

In this section we examine some of the properties of the magnetised Reissner-Nordström solution, which can be read off from our results for the magnetised Kerr-Newman solution in appendix B by setting the rotation parameter a and the magnetic charge parameter p to zero. The solution is then given by

$$\begin{aligned} d\hat{s}_4^2 &= H\,[-f dt^2 + f^{-1}\,dr^2 + r^2 d\theta^2] + H^{-1}\,r^2\sin^2\theta\,(d\phi-\omega dt)^2\,,\\ \hat{A} &= \Phi_0 dt + \Phi_3(d\phi - \omega dt)\,, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} f &= 1 - \frac{2m}{r} + \frac{q^2}{r^2}\,,\\ H &= 1 + \tfrac{1}{2}B^2(r^2\sin^2\theta + 3q^2\cos^2\theta) + \tfrac{1}{16}B^4(r^2\sin^2\theta + q^2\cos^2\theta)^2\,,\\ \omega &= -\frac{2qB}{r} + \tfrac{1}{2}qB^3\,r(1+f\cos^2\theta)\,,\\ \Phi_0 &= -\frac{q}{r} + \tfrac{3}{4}qB^2 r\,(1+f\cos^2\theta)\,,\\ \Phi_3 &= \frac{2}{B} - H^{-1}\Big[\frac{2}{B} + \tfrac{1}{2}B(r^2\sin^2\theta + 3q^2\cos^2\theta)\Big]\,. \end{aligned} \tag{2.2}$$

The scalar potentials ψ, χ and σ arising in the $SU(2,1)$ transformation procedure are given by

$$\begin{aligned} \psi &= -\frac{q\cos\theta\,[1 - \frac{1}{4}B^2(r^2\sin^2\theta + q^2\cos^2\theta)]}{H}\,,\\ \chi &= \frac{2}{B} - \frac{1}{H}\,\Big[\frac{2}{B} + \tfrac{1}{2}B(r^2\sin^2\theta + 3q^2\cos^2\theta)\Big]\,,\\ \sigma &= -\frac{qB\cos\theta}{H^2}\,\Big[q^2\cos^2\theta - \tfrac{1}{4}B^2(r^2\sin^2\theta + q^2\cos^2\theta)(r^2\sin^2\theta + 4q^2\cos^2\theta)\\ &\qquad -\tfrac{1}{16}B^4(r^2\sin^2\theta + q^2\cos^2\theta)^3\Big]\,, \end{aligned} \tag{2.3}$$

The Killing vector

$$\ell = \frac{\partial}{\partial t} - \Omega_H\,\frac{\partial}{\partial\phi} \tag{2.4}$$

becomes null on the horizon at $r = r_+$ where r_+ is the larger root of $f(r) = 0$, and where

$$\Omega_H = \frac{2qB}{r_+} - \frac{q r_+ B^3}{2} \tag{2.5}$$

is the angular velocity at the horizon. The physical electric charge $Q = 1/(4\pi)\int *F$ is given by

$$Q = q\,(1 - \tfrac{1}{4}q^2B^2)\,. \tag{2.6}$$

The magnetised Reissner-Nordström metric has an ergoregion where g_{00} becomes positive. To see this, we note that

$$g_{00} = -fH + H^{-1}\omega^2 r^2 \sin^2\theta \,. \tag{2.7}$$

Firstly, from the fact that the first term in (2.7) vanishes on the horizon while the second term contributes positively when $\sin\theta \neq 0$, it is evident that g_{00} will be positive near to the exterior of the horizon. This region is analogous to the ergoregion outside the horizon of a rotating Kerr black hole. It can also be seen that g_{00} will be positive close to the polar axes at large r with $\sin\theta$ becoming small such that $r\sin\theta$ is held fixed. To see this, it is convenient to introduce cylindrical coordinates ρ and z, defined by

$$\rho = r\sin\theta \,, \qquad z = r\cos\theta \,. \tag{2.8}$$

Making an expansion of g_{00} in inverse powers of z, we find

$$g_{00} = \frac{16B^6 q^2 (z^2 + 2mz)\rho^2}{16 + 8B^2(\rho^2 + 3q^2) + B^4(\rho^2 + q^2)^2} + \mathcal{O}(z^0) \,, \tag{2.9}$$

and therefore g_{00} can be arbitrarily large and positive at large z, holding ρ fixed. A numerical study of the metric function g_{00} reveals that the two regions of positivity described above are in fact connected. A plot showing the ergoregion for a representative example is shown in figure 1 below. The ergoregion extends to infinity near the poles regardless of how small B or q are, provided that they are non-zero. It asymptotically approaches the z axis as z tends to infinity.

Note that the ergoregion is absent if $q = 0$, in which case the metric reduces to the Schwarzschild-Melvin solution.

Although simpler than the general magnetised Kerr-Newman case, the magnetised Reissner-Nordström metric is still quite complicated. Near infinity, it resembles a much simpler, but little known, stationary vacuum metric, which is described in appendix C. That metric also exhibits an ergoregion near infinity, which is qualitatively similar to the more complicated ergoregion in the magnetised black holes. It can be obtained by starting from flat space and then acting with an $SU(2,1)$ transformation within the class described in appendix A.

2.2 Magnetised magnetic Reissner-Nordström

For completeness, we record here the expressions for the magnetised Reissner-Nordström solution carrying a magnetic, rather than electric, charge. This solution is obtained by

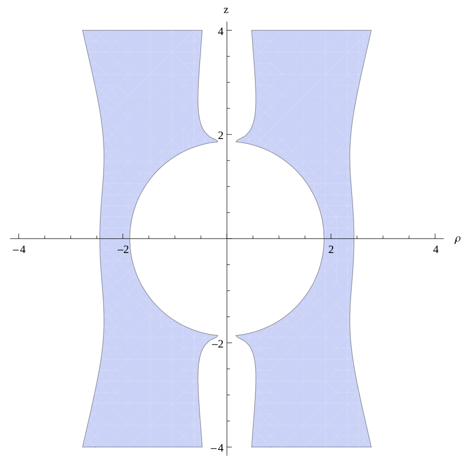

Figure 1: *The shading indicates a cross-section through the ergoregion for a magnetised Reissner-Nordström black hole, with $M = 1$, $B = 1$, $q = \frac{1}{2}$. The full ergoregion is the surface of revolution obtained by rotating this around the vertical axis. The ergoregion extends to infinity in the positive and negative z directions.*

setting $a = 0$ and $q = 0$ in the results for the magnetised Kerr-Newman solution that are obtained in appendix B.

$$\begin{aligned} ds_4^2 &= H\,[-f dt^2 + f^{-1}\,dr^2 + r^2 d\theta^2] + H^{-1}\,r^2 \sin^2\theta\,d\phi^2\,, \\ A &= \Phi_3\,d\phi\,, \end{aligned} \tag{2.10}$$

with

$$\begin{aligned} f &= 1 - \frac{2m}{r} + \frac{p^2}{r^2}\,, \\ H &= [1 + \tfrac{1}{4}B^2 r^2 \sin^2\theta - pB\cos\theta + \tfrac{1}{4}p^2 B^2 \cos^2\theta]^2\,, \\ \Phi_3 &= \frac{-p\cos\theta + \frac{1}{2}B(r^2\sin^2\theta + p^2\cos^2\theta)}{H}\,. \end{aligned} \tag{2.11}$$

3 Properties of the Magnetised Kerr-Newman Black Hole

In this section, we investigate some of the properties of the magnetised Kerr-Newman solution, which is constructed in appendix B using the appropriate $SU(2,1)$ transformations described in appendix A. For simplicity, we shall restrict attention to the case where the original seed solution is a Kerr-Newman solution carrying purely electric charge. Thus, we set $p = 0$ in all the results obtained in appendix B.

3.1 Electric and magnetic charges

If we set $p = 0$ in the magnetised Kerr-Newman solution, then the requirement of no conical deficit at the poles of the sphere implies that the azimuthal angle ϕ should have period [24]

$$\begin{aligned}\Delta\phi &= 2\pi H|_{\theta=0} = 2\pi H|_{\theta=\pi}\,,\\ &= 2\pi\Big[1 + \tfrac{3}{2}q^2B^2 - 2aqmB^3 + (a^2m^2 + \tfrac{1}{16}q^4)B^4\Big]\,. \end{aligned} \tag{3.1}$$

Thus the conserved electric charge is given by

$$Q = \frac{1}{4\pi}\int_{S^2} \hat{*}\hat{F} = \frac{\Delta\phi}{4\pi}\Big[\psi\Big]_{\theta=0}^{\theta=\pi}\,, \tag{3.2}$$

and hence

$$Q = q(1 - \tfrac{1}{4}q^2B^2) + 2amB\,. \tag{3.3}$$

Note that to obtain a neutral black hole with conserved charge $Q = 0$, we need to start with a charged rotating black hole with charge given by $q(1 - \frac{1}{4}q^2B^2) = -2amB$. In the limit that $qB << 1$ we recover Wald's result (1.3).

The conserved magnetic charge $P = 1/(4\pi)\int \hat{F}$ is equal to zero in this case where we set $p = 0$.

3.2 Ergoregions

As in the Kerr or Kerr-Newman solution itself we expect, of course, that there should be a compact ergoregion in the vicinity of the exterior of the horizon. However, in the magnetised Kerr-Newman solution the ergoregion in general extends out to infinity in the vicinity of the rotation axis. As we shall discuss, there is one exceptional circumstance where this does not occur, and that is if the charge parameter in the solution is chosen to be given by $q = -amB$. If we substitute this relation into the expression (3.3) for the electric charge Q on the black hole, we find

$$Q = amB(1 + \tfrac{1}{4}a^2m^2B^4)\,. \tag{3.4}$$

It is striking that in the small-B limit the magnitude of the charge is one half that obtained by Wald (1.3).

Consider the Killing vector field

$$K_\Omega = \frac{\partial}{\partial t} + \Omega\,\frac{\partial}{\partial\phi}\,, \tag{3.5}$$

where the angular velocity Ω is a constant which we shall choose later. If we look at large distances while holding the polar angle θ fixed, the dominant term in the large-r expansion

of $K^\mu_\Omega K^\nu_\Omega\, g_{\mu\nu}$ is negative, and has the generic form that one expects in a Melvin universe, with

$$K^\mu_\Omega K^\nu_\Omega\, g_{\mu\nu} = -\tfrac{1}{16} B^4 r^4 \sin^4\theta + \mathcal{O}(r^3)\,. \tag{3.6}$$

Thus the Killing vector K_Ω is timelike at large r, for fixed θ, for any choice of Ω. However, if we look at the region where r is large but instead $r\sin\theta$ is held fixed, then it turns out that K_Ω becomes spacelike, again for any value of Ω, signaling the occurrence of an ergoregion that extends out to infinity near the rotation axis. To see this, it is convenient to use cylindrical coordinates ρ and z, as defined in (2.8), instead of r and θ. We are then interested in probing the region where z is large while ρ remains small.

With p taken to be zero, we find that the expansion of $K^\mu_\Omega K^\nu_\Omega\, g_{\mu\nu}$ in inverse powers of z is given by

$$K^\mu_\Omega K^\nu_\Omega\, g_{\mu\nu} = \frac{16B^6(q+amB)^2\rho^2}{W}\, z^2 - \frac{4B^6(q+amB)[8qm + aB(q^2+4m^2)]\rho^2}{W}\, z + \mathcal{O}(z^0)\,, \tag{3.7}$$

where W is the positive quantity

$$W = 16 + 8B^2\rho^2 + B^4(\rho^2+q^2)^2 + 24B^2(q+\tfrac{2}{3}amB)^2 + \tfrac{16}{3}a^2m^2B^2\,. \tag{3.8}$$

Thus $K^\mu_\Omega K^\nu_\Omega\, g_{\mu\nu}$ will become large and positive in this region unless we choose $q=-amB$.

Whilst it is easy to see that there is an ergoregion near the rotation axis that extends out to infinity if $q \ne -amB$, more work is required to establish what happens if $q=-amB$. As a start, we may investigate the large-z region at fixed ρ, where we saw that K_Ω became spacelike in the previous discussion. Setting $q=-amB$ and expanding in inverse powers of z, we now find

$$K^\mu_\Omega K^\nu_\Omega g_{\mu\nu} = -\frac{F_+\,F_-}{16(4+a^2m^2B^4+B^2\rho^2)^2} + \mathcal{O}(z^{-1})\,, \tag{3.9}$$

where

$$\begin{aligned} F_\pm &= (4+B^2\rho^2)^2 + 2a^2m^2B^4(4+B^2\rho^2) + a^4m^4B^8 \\ &\quad \pm[16\Omega + 2am^2B^4(12+a^2B^2)]\rho\,. \end{aligned} \tag{3.10}$$

By choosing the angular velocity to be given by

$$\Omega = \Omega_s \equiv -\tfrac{1}{8} am^2B^4(12+a^2B^2)\,, \tag{3.11}$$

we have

$$F_+ = F_- = (4+B^2\rho^2+a^2m^2B^4)^2\,, \tag{3.12}$$

and thus

$$K^\mu_\Omega K^\nu_\Omega g_{\mu\nu} = -\tfrac{1}{16}(4 + a^2 m^2 B^4 + B^2\rho^2)^2 + \mathcal{O}(z^{-1})\,, \tag{3.13}$$

We see that with the angular velocity Ω chosen as in (3.11), the Killing vector K_Ω defined in (3.5) is timelike in this region. Thus it appears that when $q = -amB$, this Killing vector K_Ω is timelike everywhere at large distances, and so the ergoregion is now confined to the neighbourhood of the horizon.

Further insight into the significance of the angular velocity Ω_s defined in (3.11) can be obtained by introducing a comoving coordinate

$$\tilde\phi = \phi - \Omega\, t\,. \tag{3.14}$$

An examination of the metric component $g_{t\tilde\phi}$ at large z and small ρ reveals that unless $q = -amB$, it diverges linearly with z. If one restricts to $q = -amB$, one finds

$$g_{t\tilde\phi} = \frac{2(8\Omega + 12am^2B^4 + a^3m^2B^6)\rho^2}{(4 + a^2m^2B^4 + B^2\rho^2)^2} + \mathcal{O}(\frac{1}{z})\,. \tag{3.15}$$

Evidently, if one makes the choice $\Omega = \Omega_s$, then the cross term $g_{t\tilde\phi}$ vanishes to lowest order in ρ, at large z. In fact, the large z expansion now takes the form

$$g_{t\tilde\phi} = -\frac{8amB^2(4 + a^2m^2B^4)\rho^2}{(4 + a^2m^2B^4 + B^2\rho^2)^2\, z} + \mathcal{O}(\frac{1}{z^2})\,. \tag{3.16}$$

The significance of the absence of a term in ρ^2 at large z is that the Killing vector field K_{Ω_s} is locally static, that is, the "twist vector"

$$\tilde\omega^\mu_\Omega = \epsilon^\mu{}_{\nu\rho\sigma}\, K^\nu_\Omega\, \nabla^\rho K^\sigma_\Omega \tag{3.17}$$

vanishes on the axis if we choose $\Omega = \Omega_s$. Thus, the choice of azimuthal coordinate $\tilde\phi$ given by (3.14), with $\Omega = \Omega_s$, provides the best approximation to a locally non-rotating inertial frame near the axis.

It can be easily verified that $|K_{\Omega_s}|^2$ is negative everywhere at large distances, and that outside the horizon, it becomes positive only within a compact ergoregion in the neighbourhood of the horizon. This ergoregion can be thought of as a deformation, induced by the external magnetic field, of the usual ergoregion outside a Kerr or Kerr-Newman black hole.

As we shall discuss in detail in the next section, it is in fact possible to find a different choice of Killing field K_Ω that is, for sufficiently large B, timelike everywhere outside the horizon. Specifically, we do this by taking $\Omega = \Omega_H$, the angular velocity of the horizon. This is defined by the condition that K_{Ω_H} be null on the horizon. Bearing in mind that

we are setting $q = -amB$ in the Kerr-Newman seed solution, we see from (B.2) that the horizons of the magnetised black hole are located at the roots of

$$r^2 - 2mr + a^2(1 + m^2B^2) = 0\,. \tag{3.18}$$

It is therefore convenient then to express the rotation parameter a as a fraction ε of the maximum (extremal) value:

$$a = \frac{\varepsilon\, m}{\sqrt{1 + m^2B^2}}\,, \qquad 0 \le \varepsilon \le 1\,. \tag{3.19}$$

Defining then the "co-extremality parameter" $\tilde{\varepsilon}$ by $\varepsilon^2 = 1 - \tilde{\varepsilon}^2$, we see that the inner and outer horizons are located at

$$r_\pm = (1 \pm \tilde{\varepsilon})\, m\,. \tag{3.20}$$

We then find that Ω_H is given by

$$\Omega_H = \frac{(1-\tilde{\varepsilon})[8 + 4(7+3\tilde{\varepsilon})m^2B^2 - 2(1+6\tilde{\varepsilon}+\tilde{\varepsilon}^2)m^4B^4 - (1+\tilde{\varepsilon})(21+2\tilde{\varepsilon}+\tilde{\varepsilon}^2)m^6B^6]}{16m(1-\tilde{\varepsilon}^2)^{1/2}(1+m^2B^2)^{3/2}}\,. \tag{3.21}$$

If B exceeds a certain value B_+, which can be determined as the smallest positive root of a rather complicated 18th-order polynomial in B^2 that we shall not present here, then numerical studies indicate that K_{Ω_H} is timelike everywhere outside the horizon. If B is taken to lie in a range $B_- < B < B_+$, then an ergoregion of toroidal topology develops outside, and disjoint from, the horizon, in the equatorial plane.[1] As B approaches B_- from above, the toroidal ergoregion develops "lobes" that extend further and further upwards and downwards along the z direction. If B is smaller than B_-, these lobes extend all the way to infinity. The value of B_- is determined as the smallest positive root of the polynomial

$$\begin{aligned}&256(1-\tilde{\varepsilon})(1+\tilde{\varepsilon})^2\, m^6B^6 + (349 - 889\tilde{\varepsilon} + 567\tilde{\varepsilon}^2 + 243\tilde{\varepsilon}^3)\, m^4B^4\\ &+4(121 + 310\tilde{\varepsilon} + 81\tilde{\varepsilon}^2)\, m^2B^2 - 108(1-\tilde{\varepsilon}) = 0\,.\end{aligned} \tag{3.22}$$

This polynomial is determined by looking at the leading-order term in the large-z expansion of $|K_{\Omega_H}|^2$ expressed in the cylindrical coordinates ρ and z.

In figure 2 below, we present plots showing a cross-section of the location of the ergoregion for K_{Ω_H} for three representative choices of the parameters. (The full ergoregion is the

[1] Numerical studies indicate that as B approaches B_+ from below, the toroidal ergoregion contracts down to a "Saturn ring" in the equatorial plane, which finally disappears when B reaches B_+. Hence B_+ can be determined from the requirement that $H(r,\theta) \equiv |K_{\Omega_H}|^2$ and $\partial H(r,\theta)/\partial r$, evaluated in the equatorial plane $\theta = \pi/2$, should simultaneously vanish.

surface of revolution obtained by rotating the plot around the vertical axis.) In the first, B is less than B_- and the ergoregion extends to infinity. In the second, B lies between B_- and B_+, and so there is a toroidal ergoregion, disjoint from the horizon. In the third plot, B is larger but still less than B_+, and so the toroidal ergoregion has contracted.

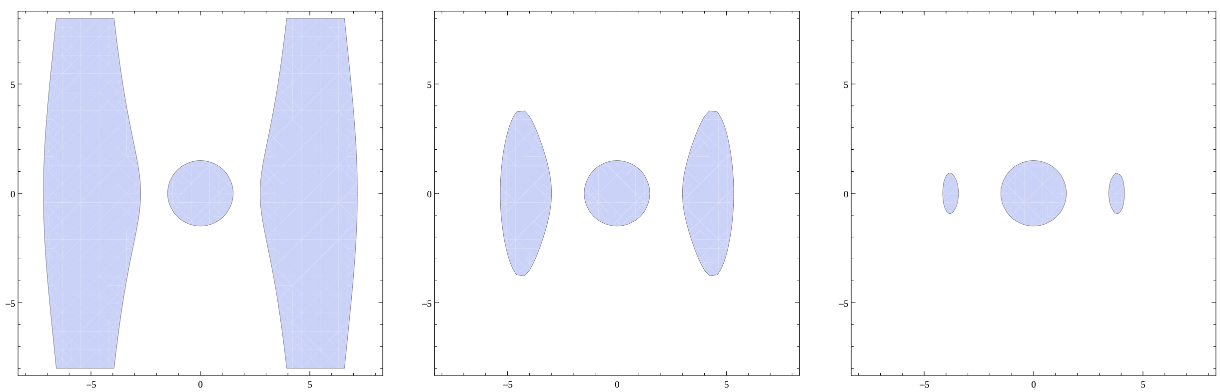

Figure 2: *The ergoregion (shaded) for the Killing vector K_{Ω_H} for three example parameter choices. In the first, $B < B_-$, and the ergoregion outside the horizon extends to infinity. In the second, $B_- < B < B_+$, and the ergoregion is a torus outside the horizon. The third plot is for a larger value of $B < B_+$, showing how the torus is contracting. (The interior of the horizon is shown shaded here too, so that its location relative to the ergoregion is evident.)*

If it should happen that Ω_H and Ω_s are equal for some choice of the parameters, then this means that the black hole has zero angular velocity as measured from the asymptotically static frame. This will occur if

$$8 + 4(7 + 3\tilde{\varepsilon})m^2B^2 - 2(13 + 18\tilde{\varepsilon} + \tilde{\varepsilon}^2)m^4B^4 - (1 + \tilde{\varepsilon})(47 + 2\tilde{\varepsilon} - \tilde{\varepsilon}^2)m^6B^6 = 0\,. \tag{3.23}$$

It is perhaps worthwhile also to comment on what happens if one simply takes the "naive" choice $\Omega = 0$ in the definition of the time-translation Killing vector K_Ω. In other words, if one simply uses the original t and ϕ coordinates of the Kerr-Newman seed solution as the time and azimuthal angle. For sufficiently small values of the magnetic field B, the Killing vector $\partial/\partial t$ is timelike everywhere outside the black hole except for a Kerr-Newman-like ergoregion near the horizon. (As usual, it is to be understood in this discussion that we are taking $q = -amB$.) As B is increased, a value $B = B_{\rm crit}$ is reached for which the ergoregion disappears altogether. This corresponds to the value of B at which the angular velocity Ω_H given in (3.21) vanishes. If B is increased beyond $B_{\rm crit}$, the ergoregion develops again around the horizon, and begins to grow "lobes" that extend upwards and downwards close to the axis of rotation. Eventually, if the magnetic field reaches or exceeds a certain

value $B_{\rm max}$, these lobes extend all the way to infinity close to the rotation axis. The value of $B_{\rm max}$ is determined by the condition that there exist a ρ such that $F_-(\rho)$ defined in (3.10) satisfy $F_-(\rho) = 0$ and $dF_-(\rho)/d\rho = 0$ simultaneously (with $\Omega = 0$). This determines that $B_{\rm max}$ is the smallest positive root of

$$64\varepsilon^6 m^{12} B^{12} - 3\varepsilon^2(36 - 16\varepsilon + 3\varepsilon^2)(36 + 16\varepsilon + 3\varepsilon^2) m^{10} B^{10} - 24\varepsilon^2(196 - 5\varepsilon^2) m^8 B^8$$
$$+16(256 + 141\varepsilon^2) m^6 B^6 + 3072(4 + \varepsilon^2) m^4 B^4 + 12288 m^2 B^2 + 4096 = 0\,. \qquad (3.24)$$

We present plots below, in figure 3, showing the ergoregion for the Killing vector $\partial/\partial t$ for two representative examples. In the first, we have $B_{\rm crit} < B < B_{\rm max}$ and prominent lobes are visible in the neighbourhood of the horizon. In the second, we have $B > B_{\rm max}$, and the lobes extend out to infinity.

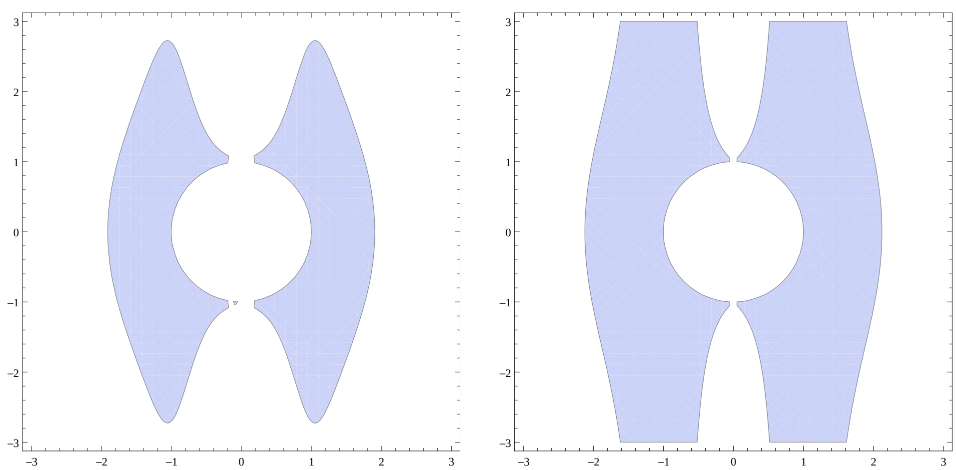

Figure 3: *The ergoregion (shaded) for the Killing vector $\partial/\partial t$ for two example parameter choices. In the first, $B_{\rm crit} < B < B_{\rm max}$ and prominent lobes have developed. In the second, $B > B_{\rm max}$ and the lobes extend to infinity.*

4 Comparison with Kerr-AdS Spacetime

In order to gain some intuition for what is happening, we shall recall some facts about rigidly rotating reference systems in asymptotically flat and asymptotically anti-de Sitter spacetimes. In flat spacetime it has been understood since early discussions of Born rigidity [25, 28, 29], the Ehrenfest paradox [26] and the Sagnac effect [27] that if one passes to a rigidly-rotating coordinate system, then it cannot be extended beyond the "velocity of light cylinder" situated at $\rho = \Omega^{-1}$, beyond which the co-rotating Killing vector $K_\Omega =$

$\partial/\partial t + \Omega\partial/\partial\phi$ becomes spacelike. Following [28, 29] we introduce a coordinate $\tilde{\phi} = \phi - \Omega t$, where t, ϕ, ρ, z are cylindrical inertial coordinates for Minkowski spacetime, in which the flat metric takes the Langevin form

$$ds^2 = -(1-\Omega^2\rho^2)\big(dt - \Omega\frac{\rho^2 d\tilde{\phi}}{1-\Omega^2\rho^2}\big)^2 + dz^2 + d\rho^2 + \frac{\rho^2}{1-\Omega^2\rho^2}d\tilde{\phi}^2\,. \tag{4.1}$$

Note that $\tilde{\phi}$ is constant along the orbits of K_Ω, i.e. $K_\Omega\tilde{\phi} = 0$, and the 3-metric

$$ds^2_\perp = dz^2 + d\rho^2 + \frac{\rho^2}{1-\Omega^2\rho^2}d\tilde{\phi}^2\,, \tag{4.2}$$

orthogonal to the orbits, is independent of time. Thus the coordinates $t, \tilde{\phi}, z, \rho$ are rigidly rotating in the sense of Born [25]. Because the twist 1-form

$$\star\,(K_\Omega)_\flat \wedge d(K_\Omega)_\flat = 2\Omega dz \tag{4.3}$$

is non-vanishing, there is no hypersurface orthogonal to the orbits of K_Ω, and the curved metric $ds^2_\perp$ is not the induced metric on any such surface. This resolves Ehrenfest's paradox [26]. The cross term in the metric , that is the term $2\Omega\rho^2 dt d\tilde{\phi}$, gives the Sagnac effect [28, 29].

For a general stationary axisymmetric spacetime with adapted coordinates t, ϕ, x^A, with $A = 1, 2$, we have

$$ds^2 = -e^{2U(x^A)}(dt + \omega(x^A)d\phi)^2 + g_{AB}(x^A)dx^A dx^B + X(x^A)d\phi^2\,. \tag{4.4}$$

The 1-form $\omega d\phi$ is the Sagnac connection [30], and if its curvature $d\omega \wedge d\phi$ is non-zero the Killing field $K = \frac{\partial}{\partial t}$ is locally rotating. Changing coordinates by setting $\tilde{\phi} = \phi - \Omega t$, where Ω is constant, gives a new metric for which

$$g_{tt} = -e^{2\tilde{U}} = -e^{2U}(1+\omega\Omega)^2 + X\Omega^2 \tag{4.5}$$

and

$$g_{t\tilde{\phi}} = -e^{2\tilde{U}}\tilde{\omega} = -e^{2U}(1+\omega\Omega)\omega + X\omega\,. \tag{4.6}$$

One cannot expect in general to be able to eliminate ω by this means, but it may be possible to make the Sagnac curvature $d\omega$ vanish along one orbit of K_ω. If so, this choice $\Omega = \Omega_s$ will define a locally static reference frame on the orbit. Of course, passing to new rigidly rotating reference system will mean that the domain of strict stationarity for which g_{tt} is negative will change. Thus, for example, a frame rotating with the angular velocity of a Kerr black hole breaks down outside an analogous velocity of light surface. Particles with

future-directed timelike momenta p_μ outside the velocity of light surface may carry negative energy with respect to the co-rotating Killing vector K_Ω, i.e. $-p_\mu K_\Omega^\mu < 0$. Thus from the point of view of a co-rotating observer, the region where K_Ω is spacelike is potentially a source of energy, i.e. it is an ergoregion. Moreover, every rotating observer has such an ergoregion. On the other hand, observers who are not rotating at infinity will find an ergoregion surrounding the black hole. In other words, the concept of an ergoregion, and its location, is observer dependent. However, there is no choice of Killing vector field that is timelike everywhere outside the horizon of a Kerr black hole, and so any observer will see an ergoregion somewhere in the exterior spacetime.

This need not, however, be the case for asymptotically anti-de Sitter (AdS) spacetimes. In AdS itself, it is possible to pass to a rotating frame in which the Killing vector $K_\Omega = \partial/\partial t + \Omega\partial/\partial\phi$ is timelike everywhere, as long as

$$\Omega^2 < \ell^{-2}\,, \tag{4.7}$$

where ℓ is the AdS radius. For this reason, when dealing with asymptotically AdS spacetimes, we need an extra criterion to decide whether or not we are in a frame that is "non-rotating at infinity." This can be done by requiring that the conformal boundary metric be non-rotating. For the Kerr-AdS black hole, the metric in this frame is given by

$$ds^2 = -\frac{\Delta_\theta\, X}{\Xi^2\, R^2}\Big(dt + \frac{2amr\sin^2\theta}{X}\, d\phi\Big)^2 + \frac{\Delta_r\, R^2\sin^2\theta}{X}\, d\phi^2 + R^2\Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big)\,, \tag{4.8}$$

where

$$\begin{aligned} \Delta_r &= (r^2+a^2)(1+r^2\,\ell^{-2}) - 2mr\,, \qquad \Delta_\theta = 1 - a^2\,\ell^{-2}\,\cos^2\theta\,, \\ X &= \Xi\,(1+r^2\,\ell^{-2})\,R^2 - 2mr\Delta_\theta\,, \qquad R^2 = r^2 + a^2\cos^2\theta\,, \qquad \Xi = 1 - a^2\,\ell^{-2}\,. \end{aligned} \tag{4.9}$$

The importance of this non-rotating frame is that questions of energy, stability and black-hole thermodynamics become much simpler and better defined [32].

In dealing with rotating black holes in anti-de Sitter backgrounds, we could of course pass to a frame that is co-rotating with respect to the black hole. One may ask whether such a frame may be extended all the way to infinity, or whether it has a velocity of light surface beyond which a co-rotating Killing vector

$$\widetilde{K}_H \equiv \frac{\partial}{\partial t} + \Omega_H\,\frac{\partial}{\partial\phi}\,, \qquad \Omega_H = \frac{a(1+r_+^2\,\ell^{-2})}{(r_+^2+a^2)} \tag{4.10}$$

becomes spacelike. If the black hole is sufficiently small that $r_+^4 < a^2\ell^2$, there is a velocity of light surface analogous to that in asymptotically flat spacetimes. On the other hand, for

black holes such that $r_+^4 > a^2\ell^2$, the Killing vector $\widetilde{K}_H$ is timelike everywhere outside the horizon. Thus in contrast to the asymptotically flat case, for sufficiently large black holes there is a choice of Killing vector field that is timelike everywhere outside the horizon, and thus it has no ergoregion.

The situation in the case of the magnetised black holes that we are considering in this paper is more involved. It is helpful to consider first the Melvin universe without a black hole. In static coordinates, the metric is

$$ds^2 = (1 + \tfrac{1}{4}B^2\rho^2)^2(-dt^2 + d\rho^2 + dz^2) + \frac{\rho^2 d\phi^2}{(1 + \frac{1}{4}B^2\rho^2)^2}\,. \tag{4.11}$$

Introducing the rotating coordinate $\tilde{\phi}$ as in (3.14), which is constant along the orbits of the Killing field $K_\Omega = \partial/\partial t + \Omega\partial/\partial\phi$, i.e. $K_\Omega(\tilde{\phi}) = 0$, the metric becomes

$$ds^2 = (1 + \tfrac{1}{4}B^2\rho^2)^2(-dt^2 + d\rho^2 + dz^2) + \frac{\rho^2(d\tilde{\phi}^2 + \Omega dt)^2}{(1 + \frac{1}{4}B^2\rho^2)^2}\,. \tag{4.12}$$

We note that the metric component $g_{t\tilde{\phi}}$ is non-vanishing, and proportional to ρ^2 when ρ is small, and that

$$g_{tt} = -(1 + \tfrac{1}{4}B^2\rho^2)^2\left(1 - \frac{\Omega^2\rho^2}{(1 + \frac{1}{4}B^2\rho^2)^4}\right). \tag{4.13}$$

From this it can be seen that if $\Omega^2 < B^2$ then the rigidly rotating Killing vector $K_\Omega = \partial/\partial t + \Omega\partial/\partial\phi$ is everywhere timelike. If $\Omega^2 > B^2$, it becomes spacelike within the annular cylinder $0 < \rho_- < \rho < \rho_+$, with $\rho_- < \rho_{\rm Melvin} < \rho_+$, where $\rho_{\rm Melvin}$ is the Melvin radius, defined by

$$\rho_{\rm Melvin} = \frac{2}{B}\,. \tag{4.14}$$

This is an indication that a general magnetised black hole solution could, for a large enough B field, be expected to have an ergoregion within an annular cylinder extending to infinity unless the coordinate system is chosen to be asymptotically static.

Turning now to the magnetised Kerr-Newman solutions, we have seen that in the general case $q \neq -amB$ the situation is much more pathological than in the Melvin universe example we have just been considering. Namely, there is *no* choice of Killing vector field, i.e. no choice of Ω in the definition (3.5), that does not have an ergoregion in the neighbourhood of infinity.

In the special case when $q = -amB$, then for given a, m and B, or equivalently a, r_+ and B, there is a range of values for Ω such that $K_\Omega = \partial/\partial t + \Omega\partial/\partial\phi$ is timelike at infinity. This range includes $\Omega = \Omega_s$, defined in (3.11), for which K_Ω is timelike at infinity for all values of a, m and B. There is also a different choice of Ω within this range, namely the

angular velocity of the horizon, $\Omega = \Omega_H(a, m, B)$, for which, provided that B is sufficiently large ($B > B_+$, defined in section 3.2), K_Ω is timelike *everywhere* outside the horizon (and lightlike *on* the horizon). This situation is analogous to what we saw in the case of AdS black holes. If, however, B is less than B_+ then for a range $B_- < B < B_+$ (with B_- defined in section 3.2), there is a toroidal ergoregion outside and disjoint from the horizon. If $B \le B_-$, this ergoregion extends to infinity near to the rotation axis.

5 Comparison with the linearised Wald analysis

It is instructive to compare our results with those that Wald obtained [1] by employing a linearised analysis starting with the Kerr solution. Wald's analysis ignored the back reaction of the magnetic field on the Kerr metric. The back reaction becomes important at radii ρ that are comparable to or greater than $\rho_{\rm Melvin}$, defined in (4.14). As long as the horizon radius is much smaller than $\rho_{\rm Melvin}$, i.e. $m << B^{-1}$, the metric at distances large compared with the horizon radius, but still much smaller than $\rho_{\rm Melvin}$, is well approximated by an asymptotically flat metric, as assumed in Wald's discussion. Therefore to make the comparison with our results, which include without approximation the non-linear effects due to the back reaction, we may linearise of our expressions for the magnetised Kerr-Newman solution (with $p = 0$). Thus, we treat q and B as small, and keep only terms up to linear order in small quantities. Stated precisely, we rescale $q \to kq$ and $B \to kB$ in the exact solution, expand up to linear order in k, and then set $k = 1$. In this approximation the metric becomes precisely the uncharged Kerr metric, and the gauge potential, after making the gauge transformation $A \to A + q/(2m)\, dt$ for convenience, becomes

$$A^{\rm lin} = -\frac{q}{2m}\, K_\flat + \tfrac{1}{2} B\, \mathfrak{m}_\flat\,, \tag{5.1}$$

Using the expression (3.3) for the physical charge Q on the black hole, which becomes, after linearisation, $Q = q + 2amB$, and using the fact that the angular momentum of the Kerr black hole is given by $j = am$, we see that (5.1) becomes

$$A^{\rm lin} = \frac{(2jB - Q)}{2m}\, K_\flat + \tfrac{1}{2} B\, \mathfrak{m}_\flat\,, \tag{5.2}$$

where $K_\flat = g_{t\mu} dx^\mu$ and $\mathfrak{m}_\flat = g_{\phi\mu} dx^\mu$. Note that j may be evaluated by a Komar integral over a surface at a radius much larger than the horizon radius, but still much smaller than the Melvin radius $2/B$. There is no obvious relation between j and a (possibly regularised) Komar integral taken over a surface whose radius is much greater than the Melvin radius.

5.1 The First and Second Laws, and injection energies

Let us suppose that for some choice of timelike Killing vector field K^μ the future-directed null generator l^μ of the horizon is given by

$$l^\mu = K^\mu + \Omega_H m^\mu \,, \tag{5.3}$$

where Ω_H is the angular velocity of the horizon. The future-directed mechanical 4-momentum p_μ of an infalling particle of mass m and charge$q$ is given by

$$p_\mu = m \frac{dx^\mu}{d\tau} \,. \tag{5.4}$$

It follows that

$$l^\mu p_\mu < 0 \,. \tag{5.5}$$

Now the canonical 4-momentum π_μ is given by

$$\pi_\mu = p_\mu + q A_\mu \tag{5.6}$$

and hence

$$l^\mu \pi_\mu - q\Phi_H < 0 \tag{5.7}$$

where $\Phi_H = -l^\mu A_\mu$ is the electrostatic potential of the horizon, and where a gauge must be chosen which is regular on the horizon. It is then known that this quantity is constant on the horizon. Now $E_p = -K^\mu \pi_\mu$ is the conserved energy (with respect to K^μ), and $J_p = m^\mu \pi_\mu$ is the conserved angular momentum of the infalling particle. Thus

$$E_p - \Omega_H J_p - \Phi_H \, q > 0 \,. \tag{5.8}$$

If we identify E_p with dE, the gain in energy of the horizon; J_p with dJ, the change in angular momentum of the horizon; and q with dQ, the change in electric charge of the horizon, we shall have

$$dE - \Omega_H dJ - \Phi_H dQ > 0 \,. \tag{5.9}$$

In the asymptotically flat case considered by Wald, the right-hand side of (5.9) equals $TdS = \frac{1}{8\pi} \kappa dA$, and we might expect this still to be true in the non-asymptotically flat case for suitable definitions of E and J.

Now for a particle falling along the axis, on which $m^\mu = 0$, we have $J_p = 0$, and so the injection energy is determined by $\Phi_H dQ$, and in Walds's case Φ_H is determined by the difference of $-A_\mu K^\mu$ between the horizon and infinity. By (5.2) this is given by

$$dQ \Big(\frac{Q}{2m} - \frac{Bj}{m} \Big) \,. \tag{5.10}$$

In our case, the analogue of Wald's injection energy would be proportional to the difference of $-A_\mu K^\mu$ evaluated on the axis between the horizon and infinity for an appropriate choice of Killing vector K^μ.

Taking $\Omega = \Omega_H$, the angular velocity of the horizon given by (3.21), we find that on the horizon,

$$(-A_\mu K^\mu_{\Omega_H})|_H = \frac{aB(1 - 2m^2B^2 - \frac{1}{4}(11 + \tilde{\varepsilon}^2)m^4B^4)}{(1 + m^2B^2)}\,. \tag{5.11}$$

At large distances, we find

$$-A_\mu K^\mu_{\Omega_H} = \tfrac{1}{2}amB^3(1 + \cos^2\theta)\,r + \mathcal{O}(r^0)\,. \tag{5.12}$$

Because $A_\mu K^\mu_{\Omega_H}$ diverges at infinity (in a direction-dependent fashion), we cannot apply Wald's injection energy argument. In his case, which we recover by setting $B^3 = 0$ in (5.12), $A_\mu K^\mu_{\Omega_H}$ tends to zero at infinity and so the difference between its value on the horizon (where it is constant) and at infinity is well defined and finite. This presumably accounts for the difference of a factor of two in his formula for the preferred charge on the hole and our formula for the value required to avoid an ergoregion in the neighbourhood of infinity.

6 Conclusion

In this paper we have resolved some longstanding puzzles concerning the behaviour of a magnetized Kerr-Newman black hole immersed in an external magnetic field B which have previously been obscured by the algebraic complexity of the exact metrics, for which we give a complete and self-contained derivation.

Specifically we have identified the criterion $Q = jB(1 + \frac{1}{4}j^2B^4)$ which must be satisfied if there is to be no ergoregion associated with the Killing vector field $K_\Omega = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}$ in the neighbourhood of infinity for some choice of angular velocity Ω.[2] If this criterion is satisfied, there are then two natural rigidly-rotating frames of reference that are of particular interest, associated with the Killing vector field K_Ω. One, which has $\Omega = \Omega_s$ (see (3.11)), may be thought of as non-rotating near infinity. In this case there is, in general, an ergoregion confined to a neighbourhood of the horizon. The other frame, for which $\Omega = \Omega_H$ (see (3.21)), may be thought of as co-rotating with the horizon. In this case, again provided our

[2]Note that Q is the conserved electric charge calculated in the exact geometry of the magnetised black hole. However, $j = am$ is the angular momentum of the original Kerr-Newman seed seed solution. It is not clear at present how one might calculate the true conserved angular momentum of the magnetised black hole. An equivalent statement of the criterion is $q = -amB$, where q is the charge parameter of the seed solution.

criterion $q = -amB$ is satisfied, numerical studies indicate there is no ergoregion at all if B is sufficiently large ($B > B_+$, defined in section 3.2): the associated Killing vector field K_Ω is everywhere timelike outside the horizon. For B in the range $B_- < B < B_+$ (where B_- is also defined in section 3.2), there is a toroidal ergoregion outside and disjoint from the horizon. If $B \le B_-$ this ergoregion extends out to infinity in a tubular region near to the rotation axis.

This somewhat non-intuitive behaviour, which is not encountered in the asymptotically flat case which is recovered in the absence of a magnetic field, may be attributed to the highly curved geometry near infinity which results from the full non-linear back-reaction, and is analogous to a similar phenomenon encountered in the case of AdS black holes.

Our criterion charge differs from the condition $Q = 2jB$ obtained by Wald on energetic grounds. This may also be attributed to the difference between the fields when back-reaction is taken into account. One may show how Wald's results can be recovered by measuring energies not with respect to infinity, but in an intermediate region whose distance is large compared with the horizon radius but small compared with the Melvin radius, $\frac{2}{B}$, at which substantial back-reaction effects set in.

The ultimate aim of the work reported here is to bring to bear the machinery of black hole thermodynamics at the fully non-linear level to the physically important problem of the energetics of black holes in magnetic fields. Our results leave many questions unanswered, but without a good understanding of the appropriate reference systems to use, progress is blocked. The results of this paper illustrate the subtlety of the problem.

Acknowledgements

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A Magnetising Transformation

We begin with the four-dimensional Einstein-Maxwell theory described by the Lagrangian

$$\mathcal{L}_4 = \hat{R} - \hat{F}^2\,. \tag{A.1}$$

(All higher-dimensional quantities are hatted.) We consider a solution with

$$\begin{aligned} d\hat{s}_4^2 &= e^{2\varphi}\, ds_3^2 + e^{-2\varphi}(dz + 2\mathcal{A})^2\,,\\ \hat{A} &= A + \chi\,(dz + 2\mathcal{A})\,, \end{aligned} \tag{A.2}$$

where all quantities on the right-hand sides are independent of z. The reduced three-dimensional Lagrangian is given by

$$\mathcal{L}_3 = R - 2(\partial\varphi)^2 - 2e^{2\varphi}(\partial\chi)^2 - e^{-4\varphi}\mathcal{F}^2 - e^{-2\varphi}F^2\,, \tag{A.3}$$

where

$$\mathcal{F} = d\mathcal{A}\,, \qquad F = dA + 2\chi\, d\mathcal{A}\,. \tag{A.4}$$

Adding Lagrange multipliers $4d\psi \wedge (F - 2\chi\mathcal{F}) + 4d\sigma \wedge \mathcal{F}$ and eliminating F and $\mathcal{F}$, we obtain the dualised Lagrangian

$$\mathcal{L}_3 = R - 2(\partial\varphi)^2 - 2e^{2\varphi}(\partial\chi)^2 - 2e^{2\varphi}(\partial\psi)^2 - 2e^{4\varphi}(d\sigma - 2\chi d\psi)^2\,. \tag{A.5}$$

The two formulations are related by

$$e^{-2\varphi}{*F} = d\psi\,, \qquad e^{-4\varphi}{*\mathcal{F}} = d\sigma - 2\chi d\psi\,. \tag{A.6}$$

This implies that

$$\hat{F} = -e^{2\varphi}{*d\psi} + d\chi \wedge (dz + 2\mathcal{A})\,. \tag{A.7}$$

The sigma model metric

$$d\Sigma^2 = d\varphi^2 + e^{\varphi}(d\chi^2 + d\psi^2) + e^{2\varphi}(d\sigma - \chi d\psi)^2 \tag{A.8}$$

is the Fubini-Study metric on the non-compact $\widetilde{\mathbb{CP}^2} = SU(2,1)/U(2)$, with $R_{ij} = -\frac{3}{2}g_{ij}$. It has the Kähler form

$$J = e^{\varphi}\,[d\varphi \wedge (d\sigma - \chi d\psi) + d\psi \wedge d\chi] = d[e^{\varphi}(d\sigma - \chi d\psi)]\,. \tag{A.9}$$

Defining the 3×3 matrices ${E_a}^b$ to have zeroes everywhere except for a 1 at row a, column b, we can parameterise a coset representative as

$$\mathcal{V} = e^{\varphi H}\, e^{-\mathrm{i}\sigma {E_0}^2}\, e^{\sqrt{2}\,\chi({E_0}^1 + {E_1}^2)}\, e^{-\mathrm{i}\sqrt{2}\,\psi({E_0}^1 - {E_1}^2)}\,, \tag{A.10}$$

where $H = {E_0}^0 - {E_2}^2$. It can be verified that $\mathcal{V}$ is in $SU(2,1)$, with

$$\mathcal{V}^\dagger \eta \mathcal{V} = \eta\,, \qquad \eta = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,, \tag{A.11}$$

with η being the invariant metric of $SU(2,1)$. Defining

$$\mathcal{M} = \mathcal{V}^\dagger \mathcal{V} \tag{A.12}$$

the Lagrangian (A.5) can be written as

$$\mathcal{L}_3 = R - \mathrm{tr}(\mathcal{M}^{-1}\partial\mathcal{M})^2\,. \tag{A.13}$$

This makes manifest that $\mathcal{L}_3$ is invariant under $SU(2,1)$, with

$$\mathcal{M} \longrightarrow \mathcal{M}' = U^\dagger \mathcal{M} U\,, \tag{A.14}$$

where U is any constant $SU(2,1)$ matrix, obeying $U^\dagger \eta U = \eta$.

The specific $SU(2,1)$ transformation that generates magnetised solutions from non-magnetised ones is given by taking

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{B}{\sqrt{2}} & 1 & 0 \\ \frac{B^2}{4} & \frac{B}{\sqrt{2}} & 1 \end{pmatrix}\,. \tag{A.15}$$

More generally, we can generate solutions with an external electric field E and magnetic field B using

$$U = \begin{pmatrix} 1 & 0 & 0 \\ \frac{(B+\mathrm{i}E)}{\sqrt{2}} & 1 & 0 \\ \frac{(B^2+E^2)}{4} & \frac{(B-\mathrm{i}E)}{\sqrt{2}} & 1 \end{pmatrix}\,. \tag{A.16}$$

Electric/Magnetic duality in four dimensions corresponds to a $U(1)$ rotation

$$\begin{aligned} \hat{F} \longrightarrow \hat{F}' &= \hat{F}\cos\alpha + \hat{*}\hat{F}\sin\alpha\,, \\ &= -e^{2\varphi} {*d\psi'} + d\chi' \wedge (dz + 2\mathcal{A})\,, \end{aligned} \tag{A.17}$$

where

$$\begin{pmatrix} \chi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix}\,. \tag{A.18}$$

At the same time, maintaining the invariance of $d\sigma - \chi d\psi$ requires transforming σ to

$$\sigma' = \sigma + \tfrac{1}{2}\sin 2\alpha\,(\psi^2 - \chi^2) - 2\sin^2\alpha\,\chi\psi\,. \tag{A.19}$$

This duality transformation is implemented on $\mathcal{M}$ by (A.14) with the $U(1) \in SU(2,1)$ matrix

$$U = \begin{pmatrix} e^{-\frac{\mathrm{i}}{3}\alpha} & 0 & 0 \\ 0 & e^{\frac{2\mathrm{i}}{3}\alpha} & 0 \\ 0 & 0 & e^{-\frac{\mathrm{i}}{3}\alpha} \end{pmatrix}\,. \tag{A.20}$$

The complex Ernst potential Φ is defined by $d\Phi = i_K(\hat{*}\hat{F} + \mathrm{i}\,\hat{F})$, where $K = \partial/\partial z$. From (A.7) and (A.17), we see that we can take

$$\Phi = \psi + \mathrm{i}\,\chi\,. \tag{A.21}$$

B Magnetised Kerr-Newman metric

We begin with the Kerr-Newman solution describing a rotating black hole carrying an electric charge q and a magnetic charge p. It is given by

$$\begin{aligned} d\hat{s}_4^2 &= -f dt^2 + R^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) + \frac{\Sigma\,\sin^2\theta}{R^2}\,(d\phi - \bar{\omega}dt)^2\,,\\ A &= \bar{\Phi}_0\,dt + \bar{\Phi}_3\,(d\phi - \bar{\omega}dt)\,, \end{aligned} \tag{B.1}$$

where

$$\begin{aligned} R^2 &= r^2 + a^2\cos^2\theta\,, \qquad \Delta = (r^2+a^2) - 2mr + q^2 + p^2\,,\\ \bar{\omega} &= \frac{a(2mr - q^2 - p^2)}{\Sigma}\,, \qquad f = \frac{R^2\Delta}{\Sigma}\,, \qquad \Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta\,, \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \bar{\Phi}_0 &= -\frac{qr(r^2+a^2)}{\Sigma} + \frac{ap\Delta\cos\theta}{\Sigma}\,,\\ \bar{\Phi}_3 &= \frac{aqr\sin^2\theta}{R^2} - \frac{p(r^2+a^2)\cos\theta}{R^2}\,. \end{aligned} \tag{B.3}$$

After applying the procedure described previously with the transformation (A.15), we arrive at the magnetised Kerr-Newman solution

$$\begin{aligned} d\hat{s}_4^2 &= H\Big[-f dt^2 + R^2\Big(\frac{dr^2}{\Delta} + d\theta^2\Big)\Big] + \frac{\Sigma\,\sin^2\theta}{H\,R^2}\,(d\phi - \omega dt)^2\,,\\ A &= \Phi_0\,dt + \Phi_3\,(d\phi - \omega dt)\,, \end{aligned} \tag{B.4}$$

where

$$H = 1 + \frac{H_{(1)}B + H_{(2)}B^2 + H_{(3)}B^3 + H_{(4)}B^4}{R^2}\,, \tag{B.5}$$

with

$$\begin{aligned} H_{(1)} &= 2aqr\sin^2\theta - 2p(r^2+a^2)\cos\theta\,,\\ H_{(2)} &= \tfrac{1}{2}[(r^2+a^2)^2 - a^2\Delta\,\sin^2\theta] + \tfrac{3}{2}\tilde{q}^2(a^2 + r^2\cos^2\theta)\,,\\ H_{(3)} &= -pa^2\Delta\sin^2\theta\cos\theta - \frac{qa\Delta}{r}[r^2(3-\cos^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)] + \frac{aq(r^2+a^2)^2(1+\cos^2\theta)}{2r}\\ &\quad -\tfrac{1}{2}p(r^4-a^4)\sin^2\theta\cos\theta + \frac{q\tilde{q}^2a[(2r^2+a^2)\cos^2\theta + a^2]}{2r} - \tfrac{1}{2}p\tilde{q}^2(r^2+a^2)\cos^3\theta\,,\\ H_{(4)} &= \tfrac{1}{16}(r^2+a^2)^2R^2\sin^4\theta + \tfrac{1}{4}ma^2r(r^2+a^2)\sin^6\theta + \tfrac{1}{4}ma^2\tilde{q}^2r(\cos^2\theta-5)\sin^2\theta\cos^2\theta\\ &\quad +\tfrac{1}{4}m^2a^2[r^2(\cos^2\theta-3)^2\cos^2\theta + a^2(1+\cos^2\theta)^2]\\ &\quad +\tfrac{1}{8}\tilde{q}^2(r^2+a^2)(r^2+a^2+a^2\cos^2\theta)\sin^2\theta\cos^2\theta + \tfrac{1}{16}\tilde{q}^4[r^2\cos^2\theta + a^2(1+\sin^2\theta)^2]\cos^2\theta\,, \end{aligned} \tag{B.6}$$

and we have defined

$$\tilde{q}^2 \equiv q^2 + p^2\,. \tag{B.7}$$

The quantity ω is given by

$$\omega = \frac{(2mr - \tilde{q}^2)a + \omega_{(1)}B + \omega_{(2)}B^2 + \omega_{(3)}B^3 + \omega_{(4)}B^4}{\Sigma}\,, \tag{B.8}$$

where

$$\begin{aligned}
\omega_{(1)} &= -2qr(r^2+a^2) + 2ap\Delta\cos\theta\,,\\
\omega_{(2)} &= -\tfrac{3}{2}a\tilde{q}^2(r^2+a^2+\Delta\cos^2\theta)\,,\\
\omega_{(3)} &= 4qm^2a^2r + \tfrac{1}{2}ap\tilde{q}^4\cos^3\theta - \tfrac{1}{2}qr(r^2+a^2)[r^2-a^2+(r^2+3a^2)\cos^2\theta]\\
&\quad +\tfrac{1}{2}ap(r^2+a^2)[3r^2+a^2-(r^2-a^2)\cos^2\theta]\cos\theta + \tfrac{1}{2}q\tilde{q}^2r[(r^2+3a^2)\cos^2\theta - 2a^2]\\
&\quad +\tfrac{1}{2}ap\tilde{q}^2[3r^2+a^2+2a^2\cos^2\theta]\cos\theta - am\tilde{q}^2(2aq+pr\cos^3\theta)\\
&\quad +qm[r^4-a^4+r^2(r^2+3a^2)\sin^2\theta] - apmr[2R^2+(r^2+a^2)\sin^2\theta]\,,\\
\omega_{(4)} &= \tfrac{1}{2}a^3m^3r(3+\cos^4\theta) - \tfrac{1}{16}a\tilde{q}^6\cos^4\theta - \tfrac{1}{8}a\tilde{q}^4[r^2(2+\sin^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)]\\
&\quad +\tfrac{1}{16}a\tilde{q}^2(r^2+a^2)[r^2(1-6\cos^2\theta+3\cos^4\theta) - a^2(1+\cos^4\theta)] - \tfrac{1}{4}a^3m^2\tilde{q}^2(3+\cos^4\theta)\\
&\quad +\tfrac{1}{4}am^2[r^4(3-6\cos^2\theta+\cos^4\theta) + 2a^2r^2(3\sin^2\theta - 2\cos^4\theta) - a^4(1+\cos^4\theta)]\\
&\quad +\tfrac{1}{8}am\tilde{q}^4r\cos^4\theta + \tfrac{1}{4}am\tilde{q}^2r[2r^2(3-\cos^2\theta)\cos^2\theta - a^2(1-3\cos^2\theta-2\cos^4\theta)]\\
&\quad +\tfrac{1}{8}amr(r^2+a^2)[r^2(3+6\cos^2\theta-\cos^4\theta) - a^2(1-6\cos^2\theta-3\cos^4\theta)]\,.
\end{aligned} \tag{B.9}$$

Since

$$\hat{F} = -e^{2\varphi}{*d\psi} + d\chi\wedge(d\phi - \omega dt)\,, \qquad \hat{*}\hat{F} = e^{2\varphi}{*d\chi} + d\psi\wedge(d\phi-\omega dt)\,, \tag{B.10}$$

the conserved electric and magnetic charges, obtained by integrating the 2-forms $\hat{*}\hat{F} = \hat{*}\hat{F}_{23}d\theta\wedge d\phi + \cdots$ and $\hat{F} = \hat{F}_{23}d\theta\wedge d\phi + \cdots$ over a 2-sphere, can be determined from the knowledge of ψ and of $\chi = \Phi_3$ respectively. In fact, comparing with the papers of Ernst et al., we find that the complex Ernst potential Φ is given by

$$\Phi = \psi + \mathrm{i}\,\chi\,. \tag{B.11}$$

In particular, we will have, in vierbein components,

$$\begin{aligned}
\hat{*}\hat{F}_{23} + \mathrm{i}\hat{F}_{23} = H_r + \mathrm{i}E_r &= \frac{f^{1/2}}{R\Delta^{1/2}\sin\theta}\frac{\partial}{\partial\theta}(\psi + \mathrm{i}\,\chi)\,,\\
-\hat{*}\hat{F}_{13} - \mathrm{i}\hat{F}_{13} = H_\theta + \mathrm{i}E_\theta &= -\frac{f^{1/2}}{R\sin\theta}\frac{\partial}{\partial r}(\psi+\mathrm{i}\,\chi)\,.
\end{aligned} \tag{B.12}$$

We find that for the magnetised Kerr-Newman solution,

$$\psi = \frac{\psi_{(0)} + \psi_{(1)}B + \psi_{(2)}B^2}{R^2H}\,, \tag{B.13}$$

where

$$
\begin{aligned}
\psi_{(0)} &= 2q(r^2+a^2)\cos\theta + 2apr\sin^2\theta\,,\\
\psi_{(1)} &= 2am[3r^2+a^2-(r^2-a^2)\cos^2\theta]\cos\theta - 2a\tilde{q}^2 r\sin^2\theta\cos\theta\,,\\
\psi_{(2)} &= -\tfrac{1}{2}q(r^2+a^2)^2\sin^2\theta\cos\theta - \tfrac{1}{2}apr(r^2+a^2)\sin^4\theta + 2a^2qmr\sin^2\theta\cos\theta\\
&\quad -apm[r^2(3-\cos^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)] - \tfrac{1}{2}q\tilde{q}^2[(r^2-a^2)\cos^2\theta+2a^2]\cos\theta\\
&\quad +\tfrac{1}{2}ap\tilde{q}^2 r\sin^2\theta\cos^2\theta\,. \qquad \text{(B.14)}
\end{aligned}
$$

The potential $\Phi_3 = \chi$ is given by

$$
\Phi_3 = \chi = \frac{\chi_{(0)} + \chi_1 B + \chi_{(2)}B^2 + \chi_{(3)}B^3}{R^2 H}\,, \qquad \text{(B.15)}
$$

where

$$
\begin{aligned}
\chi_{(0)} &= aqr\sin^2\theta - p(r^2+a^2)\cos\theta\,,\\
\chi_{(1)} &= \tfrac{1}{2}(\Sigma\sin^2\theta + 3\tilde{q}^2R^2)\,,\\
\chi_{(2)} &= \tfrac{3}{4}aqr(r^2+a^2)\sin^4\theta - \tfrac{3}{4}p(r^2+a^2)^2\sin^2\theta\cos\theta + 3a^2pmr\sin^2\theta\cos\theta\\
&\quad +\tfrac{3}{2}aqm[r^2(3-\cos^2\theta)\cos^2\theta + a^2(1+\cos^2\theta)] - \tfrac{3}{4}aq\tilde{q}^2 r\sin^2\theta\cos^2\theta\\
&\quad -\tfrac{3}{4}p\tilde{q}^2[(r^2-a^2)\cos^2\theta + 2a^2]\cos\theta\,,\\
\chi_{(3)} &= \tfrac{1}{8}R^2(r^2+a^2)^2\sin^4\theta + a^2mr(r^2+a^2)\sin^6\theta - \tfrac{1}{2}a^2\tilde{q}^2mr(5-\cos^2\theta)\sin^2\theta\cos^2\theta\\
&\quad +\tfrac{1}{2}a^2m^2[r^2(3-\cos^2\theta)^2\cos^2\theta + a^2(1+\cos^2\theta)^2]\\
&\quad +\tfrac{1}{4}\tilde{q}^2(r^2+a^2)[r^2+a^2+a^2\sin^2\theta]\sin^2\theta\cos^2\theta\\
&\quad +\tfrac{1}{8}\tilde{q}^4[r^2\cos^2\theta + a^2(2-\cos^2\theta)^2]\,. \qquad \text{(B.16)}
\end{aligned}
$$

The potential Φ_0 is given by

$$
\Phi_0 = \frac{\Phi_0^{(0)} + B\,\Phi_0^{(1)} + B^2\Phi_0^{(2)} + B^3\Phi_0^{(3)})}{4\Sigma}\,, \qquad \text{(B.17)}
$$

where

$$
\begin{aligned}
\Phi_0^{(0)} &= 4[(-qr(r^2+a^2) + ap\Delta\cos\theta]\,,\\
\Phi_0^{(1)} &= -6a\tilde{q}^2(r^2+a^2+\Delta\cos^2\theta)\,,\\
\Phi_0^{(2)} &= -3q[(r+2m)a^4 - (r^2+4mr+\Delta\cos^2\theta)r^3 + (2\tilde{q}^2(r+2m) - 6mr^2 - 8m^2r\\
&\qquad -3\Delta r\cos^2\theta)] + 3p\Delta[3ar^2+a^3+a(a^2+\tilde{q}^2-r^2)\cos^2\theta]\cos\theta\,, \qquad \text{(B.18)}\\
\Phi_0^{(3)} &= -\tfrac{1}{2}a\Big\{4a^4m^2 + a^4\tilde{q}^2 + 12a^2m^2\tilde{q}^2 + 2a^2\tilde{q}^4 + 2a^4mr - 24a^2m^3r + 4a^2m\tilde{q}^2r\\
&\quad -24a^2m^2r^2 - 4a^2mr^3 - 12m^2r^4 - \tilde{q}^2r^4 - 6mr^5 - 6r\Delta[2m(r^2+a^2) - \tilde{q}^2r]\cos^2\theta\\
&\quad +\Delta(\tilde{q}^4 - 3\tilde{q}^2r^2 + 2mr^3 + a^2(4m^2+\tilde{q}^2-6mr)]\cos^4\theta\Big\}\,.
\end{aligned}
$$

The vector potential generating the electromagnetic field is given by (B.4), using (B.8), (B.9), (B.15), (B.16), (B.17) and (B.18). In principle, this could be used to calculate the orbits of charged particles in the magnetised Kerr-Newman solution.

C Generating Taub Cosmological Metric

A "group commutator" $U(E,0)^{-1}U(0,B)^{-1}U(E,0)U(0,B)$, where $U(E,B)$ is given by (A.16), takes the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ic & 0 & 1 \end{pmatrix}, \tag{C.1}$$

where $c = EB$. Starting from the flat space metric

$$d\bar{s}_4^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{C.2}$$

and acting with the $SU(2,1)$ transformation (C.1), we obtain the Ricci-flat metric

$$ds_4^2 = (1 + c^2\rho^4)(-dt^2 + dz^2 + d\rho^2) + \frac{\rho^2}{1 + c^2\rho^4}(d\phi - 4czdt)^2, \tag{C.3}$$

where we have defined $\rho = r\sin\theta$, $z = r\cos\theta$. This is an analytic continuation of a Bianchi II cosmological metric originally obtained by Taub [31].

The metric (C.3) has

$$-g_{00} = (1 + c^2\rho^4)^{-1}\left[(1 + c^2\rho^4)^2 - 16c^2z^2\rho^2\right]. \tag{C.4}$$

Thus there is an ergoregion when

$$|4cz\rho| > 1 + c^2\rho^4. \tag{C.5}$$

References

[1] R. M. Wald, *Black hole in a uniform magnetic field,* Phys. Rev. D **10** (1974) 1680.

[2] A. Papapetrou, *Champs gravitationnels stationnaires a symetrie axiale,* Ann. Inst. Henri Poincare Phys. Theor. IV (1966) 83.

[3] A. R. King, J. P. Lasota and W. Kundt, *Black holes and magnetic fields*, Phys. Rev. D **12** (1975) 3037.

[4] G. W. Gibbons, *Black holes, magnetic fields and particle creation*, Mon. Not. Roy. Astron. Soc. **177** (1976) 37P.

[5] R. D. Blandford and R. L. Znajek, *Electromagnetic extractions of energy from Kerr black holes*, Mon. Not. Roy. Astron. Soc. **179** (1977) 433.

[6] J. Bičák, *Black holes under external influence*, Pramana **55** (2000) 481 [gr-qc/0101091].

[7] J. Bičák, V. Karas and T. Ledvinka, *Black holes and magnetic fields*, Submitted to: IAU symp. [astro-ph/0610841].

[8] S. S. Komissarov and J. C. McKinney, *Meissner effect and Blandford-Znajek mechanism in conductive black hole magnetospheres*, Mon. Not. Roy. Astron. Soc. Lett. **377** (2007) L49 [astro-ph/0702269].

[9] L. -X. Li, *Electromagnetic energy for a charged Kerr black hole in a uniform magnetic field*, Phys. Rev. D **61** (2000) 084033 [astro-ph/0001494].

[10] F. J. Ernst, *Black holes in a magnetic universe*, J. Math. Phys. 17, 54 (1976);

[11] F. J. Ernst and W. J. Wild, *Kerr black holes in a magnetic universe*, J. Math. Phys. 17, 182 (1976);

[12] D. V. Galtsov and V. I. Petukhov, *Black hole in an external magnetic field*, Zh. Eksp. Teor. Fiz. **74** (1978) 801.

[13] A. N. Aliev, D. V. Galtsov and A. A. Sokolov, *Rotating black hole in a strong magnetic field*, Sov. Phys. J. **23** (1980) 179.

[14] V. I. Dokuchaev, *A black hole in a magnetic universe* Sov. Phys. JETP **65** (1987) 1079.

[15] A. N. Aliev and D. V. Galtsov, *Effective ergospheres of magnetized black Holes and the Kerr-Newman-Ernst solution*, Sov. Phys. JETP **67** (1988) 1525.

[16] A. N. Aliev and D. V. Galtsov, *Physical interpretation of the Kerr-Newman-Ernst solution*, Sov. Phys. J. **32** (1989) 790.

[17] A. N. Aliev and D. V. Galtsov, *Exact solutions For magnetized black holes*, Astrophys. Space Sci. **155** (1989) 181.

[18] T. Damour, *Black hole eddy currents*, Phys. Rev. D **18** (1978) 3598.

[19] K. S. Thorne, (Ed.), R. H. Price, (Ed.) and D. A. Macdonald, (Ed.), *Black holes: The membrane paradigm*, (New Haven, USA): Yale Univ. PR. (1986) 367p

[20] M. A. Melvin, *Pure magnetic and electric geons*, Phys. Lett. **8** (1964) 65.

[21] E. Radu, *A note on Schwarzschild black hole thermodynamics in a magnetic universe*, Mod. Phys. Lett. A **17** (2002) 2277 [gr-qc/0211035].

[22] B. K. Harrison, *New solutions of the Einstein-Maxwell equations from old*, J. Math. Phys. 9, 1744 (1968)

[23] I. G. Moss, *Black holes with current loops revisited*, Phys. Rev. D **83** (2011) 124046 [arXiv:1102.3022 [gr-qc]].

[24] W. A. Hiscock, *On black holes in magnetic universes*, J. Math. Phys. **22** (1981) 1828.

[25] M. Born, *Die Theorie des starren Elektrons in der Kinematic des Relativitäts-Prinzipes*, *Ann Phys Lpz* **30**(1909) 11

[26] P. Ehrenfest, *Gleichförmige Rotation starrer Körpe und Relativitätstheorie*, *Phys. Zeit.* **10**(1909) 918

[27] G. Sagnac, *L'éther lumineux démontré par l'effet du vent relatif d'éther dans une interférométre en rotation uniforme*, *C.R. de l'Acad. des Sciences* **157** (1913) 708-10

[28] P. Langevin, *Sur les theories de relativité de M. Sagnac*, C.R. de l'Acad. des Sciences **173**, 831 (1921).

[29] P. Langevin, *Sur l'expérience de Sagnac*, C.R. de l'Acad. des Sciences, **205**, 304 (1937).

[30] A. Ashtekar and A. Magnon, *The Sagnac effect in general relativity*, *J. Math. Phys.* **16** (1975) 341 .

[31] A. H. Taub, *Empty spacetimes admitting a three parameter group of motions*, *Ann. Math.* **53** (1951) 472.

[32] G. W. Gibbons, M. J. Perry and C. N. Pope, *The First law of thermodynamics for Kerr-anti-de Sitter black holes*, Class. Quant. Grav. **22** (2005) 1503 [hep-th/0408217].